Method of Proofs

Support slides

Amartya Kundu Durjoy

Lecturer

CSE,UGV

Proof Techniques - direct proofs

Here's what you know:

Ellen is a math major or a CS major.

If Ellen does not like discrete math, she is not a CS major.

If Ellen likes discrete math, she is smart.

Ellen is not a math major.

Can you conclude Ellen is smart?

$M \vee C$

$\neg D \rightarrow \neg C$

$D \rightarrow S$

$\neg M$

$$((M \vee C) \wedge (\neg D \rightarrow \neg C) \wedge (D \rightarrow S) \wedge (\neg M)) \rightarrow S$$

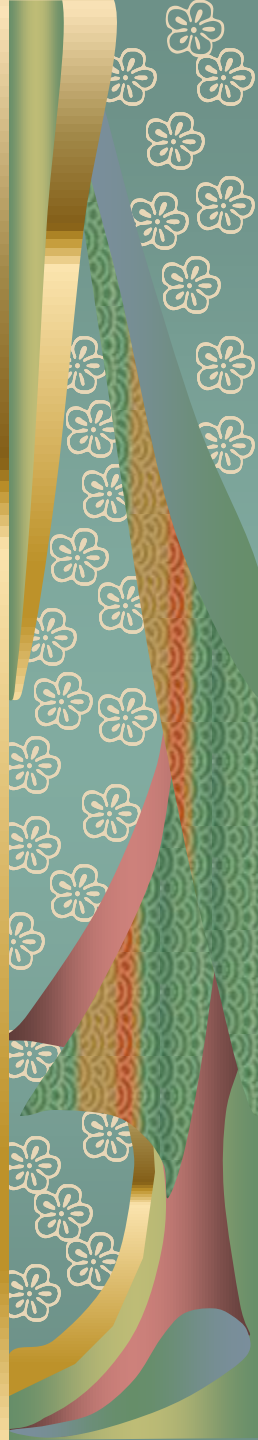
?

Proof Techniques - direct proofs

In general, to prove $p \rightarrow q$, assume p and show that q follows.

$$((M \vee C) \wedge (\neg D \rightarrow \neg C) \wedge (D \rightarrow S) \wedge (\neg M)) \rightarrow S$$

?



Proof Techniques - direct proofs

1. $M \vee C$	Given
2. $\neg D \rightarrow \neg C$	Given
3. $D \rightarrow S$	Given
4. $\neg M$	Given
5. C	DS (1,4)
6. D	MT (2,5)
7. S	MP (3,6)

Ellen is smart!

Proof Techniques - vacuous proofs

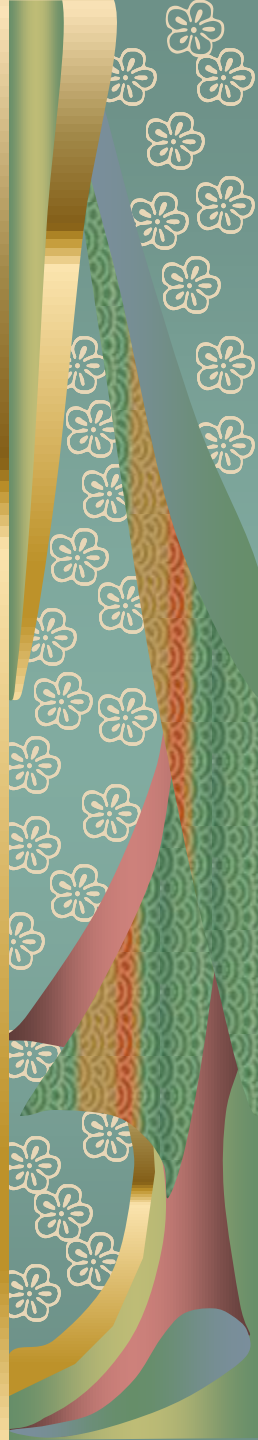
In general, to prove $p \rightarrow q$, assume p and show that q follows.

But $p \rightarrow q$ is also TRUE if p is FALSE.

Suggests proving $p \rightarrow q$ by proving $\neg p$.

Ex. p: There is good Chinese food in KUET.
q: I'll give you each $10.

Since p is FALSE, $p \rightarrow q$ is TRUE
(but we don't know a thing about q)



Proof Techniques - trivial proofs

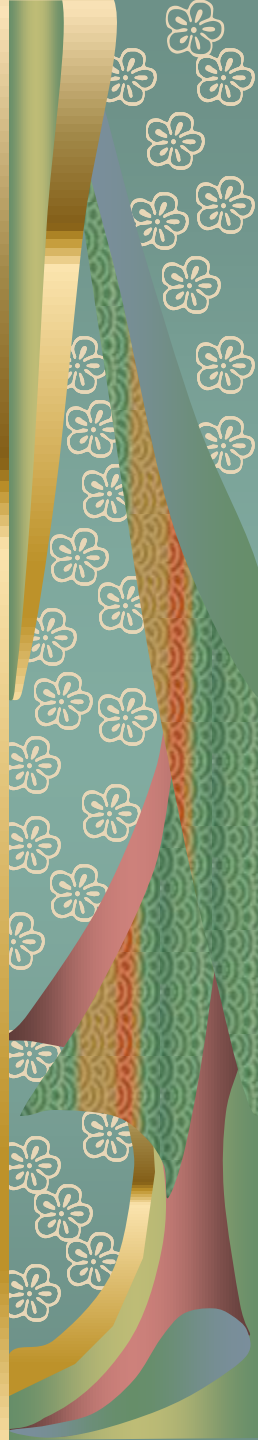
In general, to prove $p \rightarrow q$, assume p and show that q follows.

But $p \rightarrow q$ is also TRUE if q is TRUE.

Suggests proving $p \rightarrow q$ by proving q.

Ex. p: there is good Chinese food in KUET
q: I'm drinking coffee

Since q is TRUE, $p \rightarrow q$ is TRUE
(the truth or falsity of p is irrelevant)



Proof Techniques - indirect proofs

Recall that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (the contrapositive)

So, we can prove the implication $p \rightarrow q$ by first assuming $\neg q$, and showing that $\neg p$ follows.

Example: Prove that if a and b are integers, and $a + b \geq 15$, then $a \geq 8$ or $b \geq 8$.

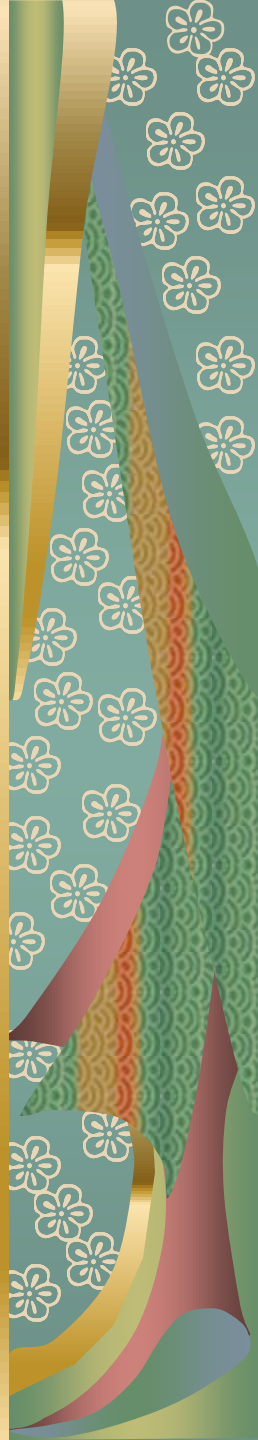
$$(a + b \geq 15) \rightarrow (a \geq 8) \vee (b \geq 8)$$

(Assume $\neg q$) Suppose $(a < 8) \wedge (b < 8)$.
(Show $\neg p$) Then $(a \leq 7) \wedge (b \leq 7)$,
and $(a + b) \leq 14$,
and $(a + b) < 15$.

Proof Techniques - proof by contradiction

To prove a proposition p, assume not p and show a contradiction.

Suppose the proposition is of the form $p \rightarrow q$, and recall that $p \rightarrow q \equiv q \vee \neg p \equiv \neg(\neg q \wedge p)$. So assuming the opposite is to assume $\neg q \wedge p$.



Proof Techniques - proof by contradiction

Example:

Rainy days make gardens grow.
Gardens don't grow if it is not hot.
When it is cold outside, it rains.

Prove that it's hot.

Given: $R \rightarrow G$
$\neg H \rightarrow \neg G$
$\neg H \rightarrow R$
Show: H

$$((R \rightarrow G) \wedge (\neg H \rightarrow \neg G) \wedge (\neg H \rightarrow R)) \rightarrow H$$
?

Proof Techniques - proof by contradiction

Given: $R \rightarrow G$

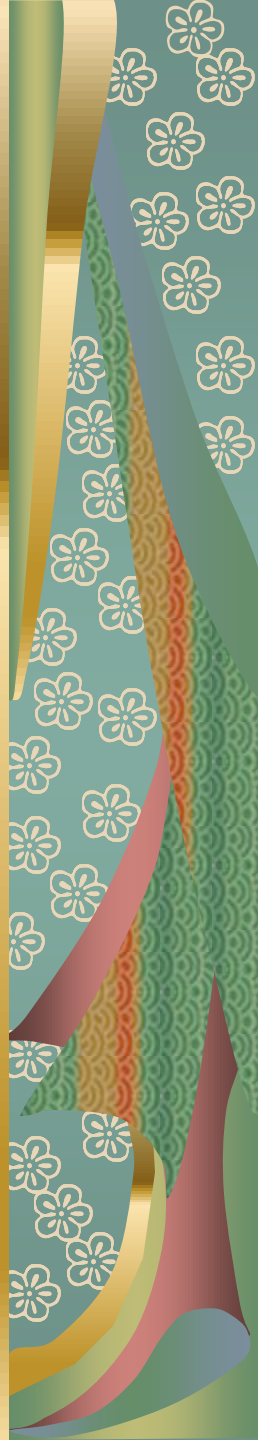
$\neg H \rightarrow \neg G$

$\neg H \rightarrow R$

Show: H

1. $R \rightarrow G$	Given
2. $\neg H \rightarrow \neg G$	Given
3. $\neg H \rightarrow R$	Given
4. $\neg H$	assume to the contrary
5. R	MP (3,4)
6. G	MP (1,5)
7. $\neg G$	MP (2,4)
8. $G \wedge \neg G$	contradiction

$\therefore H$



Proof Techniques - proof by contradiction

Classic proof that $\sqrt{2}$ is irrational.

Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = a/b$ for some integers a and b (relatively prime).

$\sqrt{2} = a/b$ implies

$2 = a^2/b^2$

$2b^2 = a^2$

a^2 is even, and so a is even (a = 2k for so

$2b^2 = (2k)^2 = 4k^2$

$b^2 = 2k^2$

b^2 is even, and so b is even (b = 2k for s

But if a and b are both even, then they are not relatively prime!

Proof Techniques - proof by contradiction

You're going to let me get away with that?

a^2 is even, and so a is even (a = 2k for some k)??

Suppose to the contrary that a is not even.

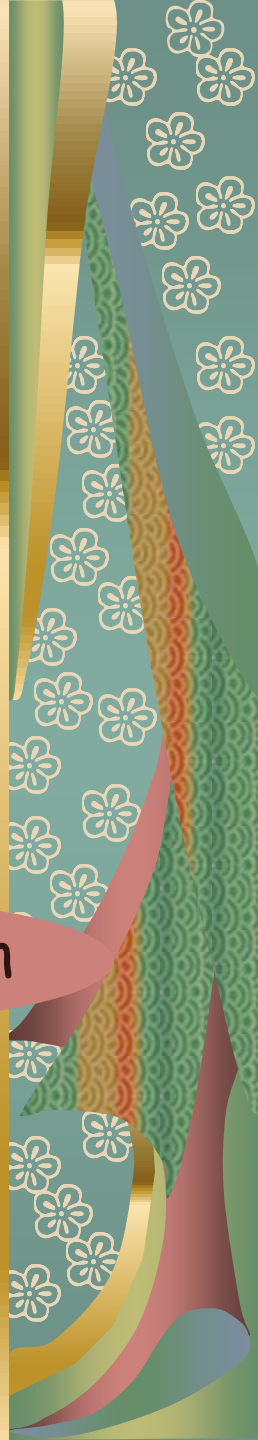
Then a = 2k + 1 for some integer k

Then $a^2 = (2k + 1)(2k + 1) = 4k2 + 4k + 1$

and a^2 is odd.

contradiction

So a really is even.



Proof Techniques - proof by cases

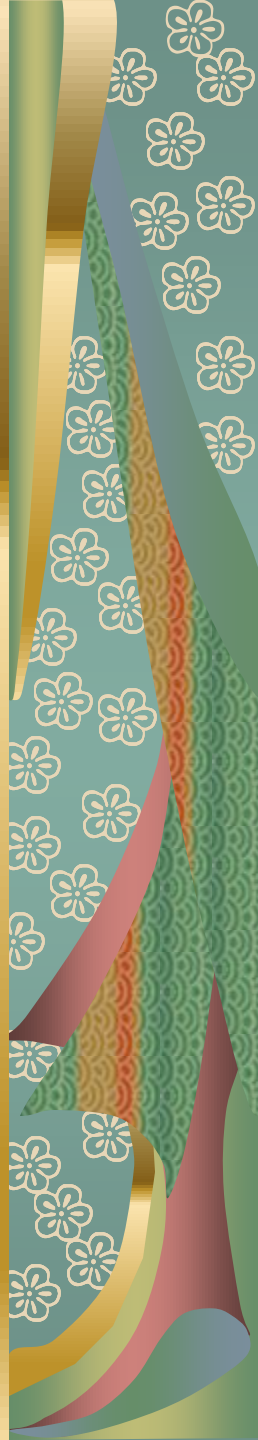
Suppose we want to prove a theorem of the form:

$p_1 \vee p_2 \vee \ldots \vee p_n \rightarrow q$

We can prove it in pieces corresponding to the cases, but which must be true?

A: $(p_1 \rightarrow q) \vee (p_2 \rightarrow q) \vee \ldots \vee (p_n \rightarrow q)$

B: $(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)$



Proof Techniques - proof by cases

$$(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)$$

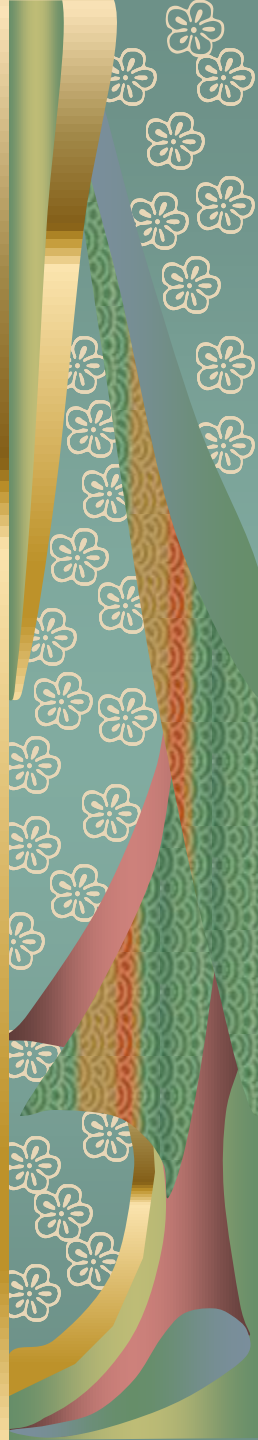
Proof for n=2:

$(p_1 \vee p_2) \rightarrow q \equiv \neg(p_1 \vee p_2) \vee q$ Defn of $\rightarrow$

$\equiv (\neg p_1 \wedge \neg p_2) \vee q$ DeMorgan's

$\equiv (\neg p_1 \vee q) \wedge (\neg p_2 \vee q)$ Distributivity

$\equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q)$ Defn of $\rightarrow$



Proofs - something for everyone…

"if x is a perfect square, and x is even, then x is divisible by 4."

Formally: $(p \wedge q) \rightarrow r$

Contrapositive: $\neg r \rightarrow \neg(p \wedge q) \equiv \neg r \rightarrow (\neg p \vee \neg q)$

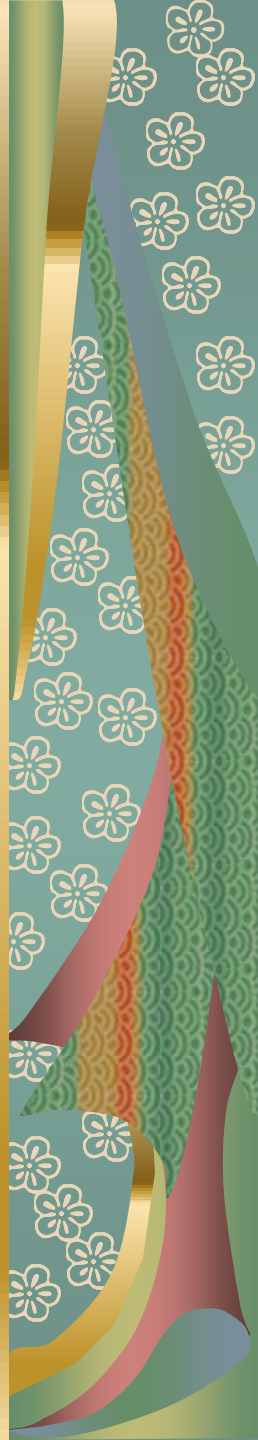
Suppose x is not divisible by 4.

Then $x = 4k + 1$, or $x = 4k + 2$, or $x = 4k + 3$.

Now structure looks like $(u_1 \vee u_2 \vee u_3) \rightarrow (\neg p \vee \neg q)$

Case 1 (&3): $x = 4k + 1$, odd, corresponds to $\neg q$

Case 2: $x = 4k + 2$, even, so must not be a perfect square.



Proofs - something for everyone…

"if x is a perfect square, and x is even, then x is divisible by 4."

Subgoal, prove Case 2:

Case 2: $x = 4k + 2$, even (so we have to show not square).

But $x = 4k + 2 = 2(2k + 1)$

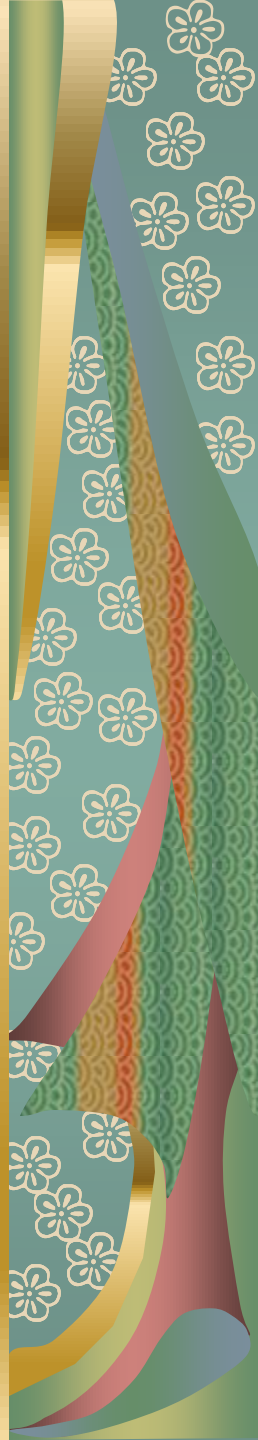
x is the product of 2 and an odd number.

So, x is not a perfect square.

Proofs - something for everyone…

If Boris becomes a pastry chef, then if he gives in to his desire for chocolate mousse, then his waistline will suffer. If his waistline suffers, then his dancing will suffer. Boris gives in to his desire for chocolate mousse. However, his dancing will not suffer. Prove that Boris does not become a pastry chef.

a) I could have done this on my own.
b) I worked it out with my partner, but I couldn't have done it alone.
c) My partner and I couldn't do it.



Proof Techniques-Quantifiers: Existence Proofs

Two ways of proving $\exists x\, P(x)$.

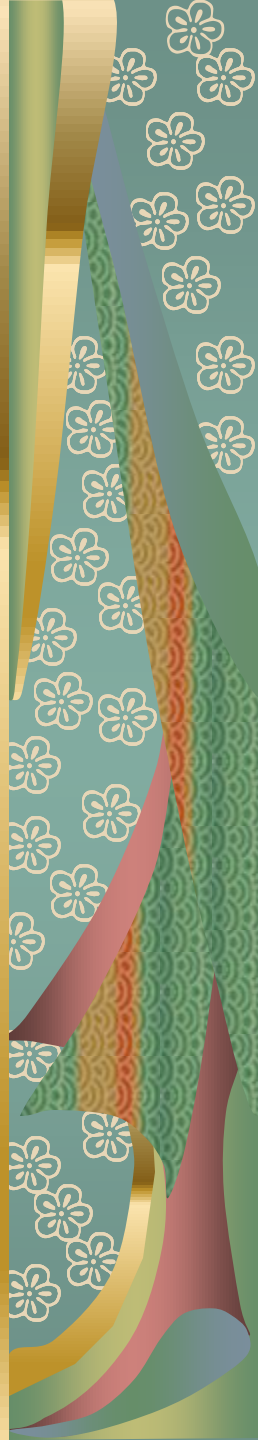
Either build one, or show one *can* be built.

Constructive

Non-constructive

Two examples, both involving n!

For the examples, think of n! as a list of factors.



CONSTRUCTIVE

Proof Techniques-Quantifiers: Existence Proofs

Example: Prove that for all integers n, there exist n consecutive composite integers.

Composite = not prime

$\forall n$ (integer), $\exists x$ so that $x, x+1, x+2, \ldots, x+n-1$ are all composite.

Proof: Let n be an arbitrary integer.

$(n + 1)! + 2$ is divisible by 2, $\therefore$ composite.

$(n + 1)! + 3$ is divisible by 3, $\therefore$ composite.

$\vdots$

$(n + 1)! + (n + 1)$ is divisible by $n + 1$, $\therefore$ composite.

$x = (n + 1)! + 2$

Proof Techniques-Quantifiers: Existence Proofs

Example: Prove that for all integers n, there exists a prime p so that $p > n$.

Infinitely many primes!

$\forall n$ (integer), $\exists p$ so that p is prime, and $p > n$.

Proof: Let n be an arbitrary integer, and consider $n! + 1$. If $(n! + 1)$ is prime, we are done since $(n! + 1) > n$. But what if $(n! + 1)$ is composite?

If $(n! + 1)$ is composite then it has a prime factorization, $p_1 p_2 \ldots p_n = (n! + 1)$

Consider the smallest p_i, how small can it be?

NON-CONSTRUCTIVE

Proof Techniques-Quantifiers: Existence Proofs

$\forall n$ (integers), $\exists p$ so that p is prime, and $p > n$.

Proof: Let n be an arbitrary integer, and consider $n! + 1$. If $(n! + 1)$ is prime, we are done since $(n! + 1) > n$. But what if $(n! + 1)$ is composite?

If $(n! + 1)$ is composite then it has a prime factorization, $p_1 p_2 \ldots p_n = (n! + 1)$

Consider the smallest p_i, and call it p. How small can it be?

Can it be 2?
Can it be 3?
Can it be 4?
Can it be n?

So, $p > n$, and we are done. BUT WE DON'T KNOW WHAT p IS!!!